Physical interpretation of the mathematical theory of wave generation by wind

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In the light of accumulated evidence in favour of Miles's mathematical theory of wave generation by wind, the author has thought it desirable to translate the theory into the form of a physical argument, which goes as follows. Travelling water waves in a wind produce, to a first approximation, airflow undulations with pressures least over crests and greatest over troughs. Hence, just above the 'critical height' where the airflow component in the direction of propagation equals the wave velocity, air after slowly overtaking a crest is turned back by the higher pressure near the trough, moves down to a lower level and back towards the crest. Similarly, behind crests, an upward movement at the critical height occurs. Quantitatively, the vertical velocity v is such that the 'vortex force' $-\rho\omega v$ (where ρ is density and ω vorticity) balances the sinusoidal pressure gradient. Furthermore, since in turbulent boundary layers vorticity decreases with height, any downflow produces a local vorticity defect, and upflow a local vorticity excess, and hence the vortex force varies about a negative mean at the critical height (although at other levels, where air moves sinusoidally, with vertical displacement and velocity 90° out of phase, the mean vortex force is zero). The negative total mean force extracts wind energy, and transfers it to the wave, enabling it to grow exponentially. For pressure gradients adequate to initiate substantial energy transfer, the critical height must be fairly small compared with the wavelength, and waves can grow whenever their velocity and direction satisfies this condition, a conclusion supported by measurements (Longuet-Higgins 1962) of the directional spectrum of wind-generated waves.

1. Introduction

The three elements needed for determining the mechanism of water-wave generation by wind, namely, a correctly argued fluid-mechanical theory, a body of comprehensively instrumented and recorded experiments, and a demonstration of adequate agreement between the two, were shown in the review by Ursell (1956) to be all wanting, but in that by Longuet-Higgins (1962) to have been in large measure supplied during the intervening six years.

The theory is due to Miles (1957, 1959a, b, 1960) and Phillips (1957, 1958, 1959, 1960). We concentrate here on the mechanism of primary transfer of energy from wind to water, rather than on secondary transfers of energy between water-wave modes (Phillips 1960), interactions with water turbulence (Phillips 1959), or the effect of wave breaking in limiting the high-frequency spectrum to an inverse-fifth-power law (Phillips 1958).

The theory of the primary transfer is based on two models (Phillips 1957; Miles 1957) which were extended and combined by Miles (1959*a*, *b*, 1960) himself, and further illuminated by Brooke Benjamin (1959).

Phillips (1957) discussed the pressure fluctuations associated with turbulent airflow over a flat surface. He greatly overestimated their root mean square, but conjectured correctly, as later measurements have shown (Willmarth 1959), that a spatial pattern of pressure variation travels over the surface at a 'pressure-pattern convection velocity' U_p, varying as it travels much more slowly than does the pressure at a fixed point which the pattern crosses. He showed that water waves with wavelength λ characteristic of such a spatial pattern, and with associated wave velocity c (which, of course, is $\sqrt{(g\lambda/2\pi)}$), would be resonantly generated if their direction of propagation and that of convection of the pattern made an angle θ satisfying

$$U_p \cos\theta = c. \tag{1}$$

By contrast, the first model due to Miles (1957) was purely two-dimensional. He showed that an exponential build-up of water waves of velocity c could occur if the mean airflow velocity U in the direction of wave propagation varied with height y above the water surface in such a way that $U''(y)$ was negative. The turbulence plays no essential part in this model, other than in creating the 'logarithmic' type of mean wind velocity profile, for which this condition is strongly satisfied. The energy transfer to the water is then substantial provided that the 'critical' height $y = y_c$, where

$$U(y_c) = c, \tag{2}$$

is small enough compared with the length λ of a wave to prevent the factor $\exp(-4\pi y_c/\lambda)$, which appears in Miles's expression for energy transfer, from being too small.

Miles's model neglects the squares of disturbances to the sheared airflow, and the dissipative or other effects on those disturbances of both viscosity and turbulence. It can be simply adapted (Miles 1960) to deal with the generation of waves at a non-zero angle θ to the wind. On these assumptions, in fact, energy can be fed into a water-wave mode only by means of components of airflow velocity in the direction of wave propagation. Hence the velocity profile $U(y)$ in Miles's original model must be replaced by $U(y)\cos\theta$, and the definition (2) of the critical height becomes

$$U(y_c)\cos\theta = c, \tag{3}$$

in close analogy with Phillips's equation (1).

Phillips's resonance mechanism by itself fails to explain the rate at which wind generates waves, partly because the pressure fluctuations due to turbulence in an airflow are altogether too weak, much weaker than Phillips (1957) estimated. Miles (1960) shows, however, that they can be regarded as providing the initial wavy disturbance of the water surface, which should then grow exponentially by his shear-flow mechanism of energy transfer, rather than linearly by Phillips's forced-vibration mechanism.

More recently, a very useful set of experimental data was obtained by the National Institute of Oceanography (Longuet-Higgins, Cartwright & Smith 1962), using simultaneous records of vertical displacement and attitude of an instrumented buoy to determine the coefficients of 1, $\cos\theta$, $\sin\theta$, $\cos 2\theta$ and $\sin 2\theta$ in the Fourier expansion with respect to θ of the 'directional spectrum' $F(\sigma, \theta)$ of the surface waves. Longuet-Higgins (1962) defines this directional spectrum so that $\rho_w gF(\sigma, \theta)\, d\sigma\, d\theta$ (where ρ_w is the density of water) is the potential energy, per unit area of water surface, of that component of the surface deformation which has frequency between σ and $\sigma + d\sigma$ and direction of propagation between θ and $\theta + d\theta$. From these five Fourier coefficients he infers an approximate r.m.s. angle between the direction of wave propagation and that of the wind, and in his figure 8 compares this angle ψ for different values of σ with $\cos^{-1}(c/U_w)$, where U_w is the wind velocity at a height of 1 wavelength (plain curve) or 0·2 wavelengths (lower broken line). Better agreement is obtained in the latter case, for wavelengths up to 100 m (longer waves than this being probably unrelated to local wind conditions), as would be expected from Miles's formula (3), according to which U_w should be $U(y_c)$, where $\exp(-4\pi y_c/\lambda)$ must not be too small. Furthermore, the closeness of agreement appears to rule out any mechanism of wave generation, such as the 'sheltering hypothesis' of Jeffreys (1925), whose effectiveness would increase indefinitely with the ratio $(U_w \cos\psi)/c$, of wind velocity component in the direction of wave propagation to wave velocity. Rather, increase of U_w/c forces the directional distribution of wave energy to an increasingly 'two-peaked' form, with peaks at angles of about $\cos^{-1}(c/U_w)$ to the wind direction.

In addition, simultaneous records of buoy elevation and air pressure showed that the phase difference between these remains close to 180°, as Miles's theory predicts, the ratio of their amplitudes also being in agreement with the theory. Longuet-Higgins (1962) found that any random component in the mean square air pressure (that is, a 'turbulent' component, as opposed to one produced directly by heaving of the water surface into the wind) was at least two orders of magnitude smaller.

Unfortunately, the instrumental technique was not accurate enough to show that the phase lag of pressure behind surface elevation was systematically a little less than 180°, as the theory predicts; but this cannot fail to be so if energy is transferred from wind to wave. To sum up, the experimental checks already described, coupled with the soundness of Miles's assumptions and calculations, give a substantial degree of confidence that the correct explanation has at last been found.

It is worth noting, no doubt, that at much higher wind velocities, as Miles (1959*b*) shows, the part of the pressure fluctuation in perfect antiphase with the surface elevation could be so great as to annul the stability of the surface (given to it by gravity and surface tension) in relation to disturbances of small wavelength; this is an instability (as opposed to a wave-generation mechanism) which would be possible even for uniform (unsheared) airflow, as Kelvin (1871) found. The importance of this 'stiffness reduction' mechanism (which first reduces frequency, i.e. wave velocity for a given wavelength, and finally gives negative stiffness, i.e. instability) must be greatly limited by the considerations of Phillips (1958) regarding effects of wave-breaking at the small wavelengths for which it operates, and (more important) by the fact that Miles's wave-generation mechanism, a 'negative damping', normally operates much sooner; although Miles (1959*b*) used experiments by Francis (1954, 1956) to demonstrate that, in liquids so viscous as to provide enough counteracting positive damping, the surface does break up only at the wind velocity at which this stiffness vanishes.

By contrast, the present paper is concerned only with that small component of surface air pressure fluctuation whose phase is 90° behind, and whose amplitude is proportional to, that of the surface elevation, so that it is able to feed energy into the water waves and cause them to grow exponentially. This component would be zero according to Kelvin's calculation for a uniform wind; Miles's crucial new discovery was that for a sheared wind of profile $U(y)$ it is positive provided that $U''(y)$ is negative.†

In one sense no further explanation is necessary: Miles's arguments are, mathematically, fully convincing. Many readers will agree, however, that if, as now appears likely, this mechanism is basically responsible for the existence of ocean waves, it is desirable to translate the mathematical arguments into statements using only the basic physical ideas of fluid mechanics. Such physical interpretations, if accurately carried out, are often useful, for example, to suggest how far a conclusion may remain valid in conditions different from those assumed in its mathematical demonstration. A physical argument is evolved and used in this way in §§ 2, 3 and 4, and summarized in § 5.

2. Properties of vorticity in two-dimensional flow

Since spectral and directional analysis reduces the general problem to that of finding how a sheared airflow over water is perturbed by the heaving motions in a harmonic train of waves propagated in the direction of the wind, and since the mechanism of energy transfer from wind to wave in this 'two-dimensional flow' problem will be explained by means of arguments mainly about vorticity, it is convenient to recall first certain properties of vorticity in two-dimensional flows. In these, the velocity has only x- and y-components (u and v), which are independent of z, so that the vortex lines are all parallel to the z-axis and are incapable of stretching; accordingly, the vorticity,‡

$$\omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}, \tag{4}$$

is conserved for each particle of fluid—except in so far as viscosity or turbulence acts to diffuse it.

The physical idea related to vorticity that will be principally used is that of 'vortex force', an effective force on unit volume of fluid equal to density times

† Actually, Brooke Benjamin (1959) showed that the pressure component in antiphase with the surface acceleration could also, through phase shifts associated with viscous action in the thin oscillatory-flow boundary layer near the surface, create a supplementary pressure component with phase 90° behind that of the surface elevation, but that this mechanism (like Kelvin's) would take effect much more gradually (as the wind velocity increased) than Miles's.

‡ Here expressed as a scalar which, strictly, is the *component* of vorticity in the negative z-direction.

vorticity times velocity, but directed at right angles to the velocity, so that its components are

$$(-\rho\omega v,\ +\rho\omega u). \tag{5}$$

Prandtl (1918) showed how this expression for vortex force generalizes the law that lift on a wing in a wind of velocity U is ρUK per unit length, where K is the circulation round it.

The precise basis of (5) is an expression for 'local rate of change' of momentum, that is, rate of change of the momentum per unit volume at a fixed point in space. If viscous or turbulent stresses are neglected this local rate of change is due partly to pressure gradient and partly to convection and can be written as the force (5) minus the gradient of the total pressure,

$$p_{\text{tot}} = p + \tfrac{1}{2}\rho(u^2+v^2), \tag{6}$$

in an equation whose x- and y-components are

$$\rho\frac{\partial u}{\partial t} = -\rho\omega v - \frac{\partial p_{\text{tot}}}{\partial x}, \tag{7}$$

$$\rho\frac{\partial v}{\partial t} = \rho\omega u - \frac{\partial p_{\text{tot}}}{\partial y}. \tag{8}$$

Equations (7) and (8) include the statement that, in steady, irrotational flow, p_{tot} has the same value everywhere. We may perhaps note, too, that steady rotational flow can be rendered locally irrotational by use of a frame of reference rotating with angular velocity $\frac{1}{2}\omega$, but only at the expense of introducing the Coriolis force $(-\rho\omega v, \rho\omega u)$, exactly as in (7) and (8).

Equation (7) will be used both in § 3 and in § 4. It may be objected that this use of a mathematical equation is inconsistent with the claim to have made an interpretation in terms of physical ideas! However, the physical ideas of total pressure and vortex force must be admitted to play crucial roles in incompressible aerodynamics, both being derived from and receiving their accurate expression in equations (7) and (8), which indeed are nothing more than rewritten forms of the basic momentum equation of the fluid. With these, as with all the most valuable physical ideas, the mathematical apparatus underlying their evolution can (if they are fully absorbed) be dispensed with, but remains useful to check that they have been used correctly!

3. Physical mechanism of energy transfer from wind to wave

We consider a sheared airflow, with velocity $U(y)$ (in the x-direction) at a height y above the mean water surface. If waves on this surface travel in the x-direction with wave velocity c, perturbing the airflow, we wish to find the energy E transferred from wind to wave per unit horizontal area, by air-pressure variations at the surface having a component in phase with the surface rate of sinking, $-\partial\eta/\partial t$ (where η is surface elevation).

It is worth noting, as Stewart (1961) has pointed out, that such pressures must also transfer momentum from wind to wave, at a rate E/c per unit area, This is clear, either because $(1/c)(-\partial\eta/\partial t) = \partial\eta/\partial x$, or because a water wave always carries energy and momentum in a ratio equal to the wave velocity c (Lamb 1932, pp. 370 and 419). The airflow must therefore lose energy and momentum at the rates E and E/c per unit horizontal area, respectively; hence, since any losses of energy and momentum at height y are approximately in the ratio $U(y)$, the level $y = y_c$, where

$$U(y_c) = c, \tag{9}$$

must be a kind of average level for loss of air energy and momentum.

Actually, both are lost almost entirely in a thin layer, centred on this level $y = y_c$. To prove this, we begin from the equation (7) for rate of change of air momentum, which simplifies when averaged over a wavelength to

$$\rho \frac{\partial \bar{u}}{\partial t} = -\rho \overline{\omega v}, \tag{10}$$

where the bars signify the average over a wavelength; the average of $\partial p_{\text{tot}}/\partial x$ is zero, because the values of p_{tot}, like other quantities in the airflow, are periodic, repeating themselves every wavelength.

Now, the vorticity ω takes the value $U'(y)$ in the unperturbed shear flow, a value decreasing with height if, as Miles assumes, $U''(y) < 0$. In the undulations of the airflow, caused by the heaving motions of the water, variations of ω from this unperturbed value $U'(y)$ are due mainly to displacements of level, since, as noted in § 2, ω is conserved for a fluid element except in so far as it is diffused by viscous or turbulent action; if these are neglected, the vorticity at (x, y), where the fluid has suffered a vertical displacement h from its unperturbed level,† is

$$\omega = U'(y-h) \doteqdot U'(y) - hU''(y). \tag{11}$$

Substituting this expression for ω in equation (10), we note that the first term, $U'(y)$, makes a zero contribution to the right-hand side of (10), since the mean of the vertical velocity v at any level must be zero by conservation of mass. Hence, the mean rate of change of momentum is

$$\rho \frac{\partial \bar{u}}{\partial t} \doteqdot \rho U''(y) \overline{hv}, \tag{12}$$

an equation previously derived by Taylor (1915).

Now, the airflow pattern must travel along with the same wave velocity, c, as the pattern of surface heaving which produces it. Hence, air at levels y other than y_c (that is, air with unperturbed velocity $U(y) \neq c$) suffers approximately sinusoidal displacements, with frequency

$$\frac{U(y)-c}{\lambda} \quad (\text{where } \lambda = \text{wavelength}); \tag{13}$$

in such sinusoidal motions, the right-hand side of (12) would be zero, since the vertical displacement h and the vertical velocity v would be 90° out of phase. (Indeed, for any periodic motions, the mean product of h and $\dot{h}$ would be zero.)

This conclusion at levels $y \neq y_c$ is not confirmed, however, when $y = y_c$. At this critical level, within the framework of small-perturbation theory, a fluid particle

† In the modified theory given at the end of §3, turbulence is taken into account approximately by making h the vertical displacement during the immediately preceding interval of duration T, where T is a sort of 'randomization time'.

travels along at the same speed as the wave, so that an ascending particle can continue to ascend, or a descending particle to descend, and the product $\overline{hv}$ in (12) increases without limit. Expressed more physically, the mean vortex force (product of vorticity and vertical velocity) is large at the critical level because rising fluid continues to rise and so to possess vorticity that is greater and greater relative to its surroundings, while conversely the vorticity of falling fluid becomes less and less relative to its surroundings.

The last two paragraphs indicate a sort of delta-function behaviour for $\overline{\omega v}$, infinite at the critical layer and zero elsewhere. A short calculation establishes the coefficient of the delta function: for a particle of fluid at height y, if $v_0(y)$ represents the amplitude of its vertical velocity fluctuations with frequency (13), we have, say,

$$v = v_0(y)\cos\left\{2\pi\frac{U(y)-c}{\lambda}t\right\}, \quad h = \frac{v_0(y)\,\lambda}{2\pi\{U(y)-c\}}\sin\left\{2\pi\frac{U(y)-c}{\lambda}\right\}, \tag{14}$$

and (12) becomes

$$\rho\frac{\partial\overline{u}}{\partial t} = \tfrac{1}{4}\rho\lambda\frac{U''(y_c)}{U'(y_c)}v_0^2(y_c)\,\delta(y-y_c), \tag{15}$$

where the formula

$$\lim_{t\to\infty}\frac{\sin zt}{z} = \pi\delta(z) \tag{16}$$

(see, for example, p. 29 of Lighthill 1958) has been used. For negative $U''(y_c)$, equation (15) implies that air momentum is lost at the critical layer at a rate E/c per unit area, where the energy transfer rate E is given by

$$E = \tfrac{1}{4}\rho\lambda c\left\{-\frac{U''(y_c)}{U'(y_c)}\right\}v_0^2(y_c), \tag{17}$$

in precise agreement with Miles (1957).

However, the determination of precise coefficients was not the object of this analysis, which is aimed rather at showing how a vertical velocity fluctuation at the critical layer (with non-zero amplitude $v_0(y_c)$) generates a concentrated vortex force, retarding the fluid in the layer, and making it give up energy and momentum to the water waves. We shall see in § 4 how the magnitude of $v_0(y_c)$ is determined by simple pressure-gradient considerations. In the meantime, it is useful to inquire how far the physical arguments may give results that are valid independently of assumptions such as Miles's neglect of the squares of perturbations or of the effects of turbulence. The answer seems to be that such non-linear effects broaden the layer of concentrated vortex force, without, however, changing its overall strength. The arguments are somewhat as follows.

At the critical layer, the greatest value of the upflow is $v_0(y_c)$ (its general value being this times $\cos\{2\pi(x-ct)/\lambda\}$). Fluid with this upflow velocity passes through the layer, supposed of thickness δ, in a time $t_c = \delta/v_0(y_c)$. Now, the variation in unperturbed velocity across the layer, approximately $U'(y_c)\,\delta$, must horizontally separate different parts of it in the available time t_c by at most some fraction α of the wavelength, say

$$\frac{U'(y_c)\,\delta^2}{v_0(y_c)} = \alpha\lambda, \tag{18}$$

if air is to retain an upflow velocity near the maximum throughout its time in the layer. Also, the total vortex force on the layer, per unit area, is some fraction β of the excess vorticity, $U''(y_c)\,\delta$, due to passage through the layer, times $v_0(y_c)$, times the thickness δ, times the density ρ, and hence, by (18), is

$$\alpha\beta\frac{U''(y_c)}{U'(y_c)}v_0^2(y_c)\rho\lambda, \tag{19}$$

agreeing with (15) if $\alpha\beta = \frac{1}{4}$.

Turbulence seems to assist the process just described, by ensuring that fluid passing upwards through the layer has on the average recently acquired the vorticity of its surroundings (by diffusion), instead of being fluid that at an earlier stage had made a similar passage downwards through the layer. Admittedly, intense turbulent fluctuations may restrict the time spent in steady upward movement through the layer to an effective average time T less than t_c. However, this would simply cause the effective critical layer to be wider,† since equation (18) must be replaced by $TU'(y_c)\,\delta = \alpha\lambda$; this would balance the smaller excess vorticity generated by fluid travelling, not right across the layer, but at most a distance $v_0(y_c)\,T$ in the time T, and the result (19) for total vortex force would remain unaltered.

4. Physical estimation of the oscillations of vertical velocity at the critical layer

In order to be able to use the formula (17) for energy transfer, Miles had to calculate $v_0(y_c)$, the amplitude of the oscillations of vertical velocity at the critical layer, first by an approximate (Miles 1957) and then by an exact (Miles 1959*a*) solution of the small-perturbation equations. A simple physical argument, again using vortex force, may be used to obtain Miles's approximate value of $v_0(y_c)$.

This starts from a first approximation $v_0(y) = \theta_0\{U(y)-c\}\,e^{-2\pi y/\lambda}$ (where θ_0 is maximum slope of the water surface; on this approximation $v_0(y_c)$ would be zero), and goes on to a second approximation with $v_0(y_c)$ non-zero. Brooke Benjamin (1959) discussed these approximations of Miles (1957) very fully, and observed that, together with a third approximation, they were given also in a paper on a totally different subject by Lighthill (1957) in the same number of the same journal!

The physical basis of the approximations appears most clearly in a frame of reference moving with the waves, in which the airflow may be regarded as steady,‡ with unperturbed velocity

$$V(y) = U(y)-c. \tag{20}$$

† Exactly the same argument applies in another interesting case, when the water was initially at rest and has been heaving only for a limited time T (less than t_c). Similarly, Miles's mathematical solution is of a type known to require, for its correctness, either some diffusive effect or a limited period of action of the forcing term.

‡ More precisely, the Miles method is to suppose in this way that a steady perturbation were set up, but then to show (§3) that it would necessarily involve energy transfer (of the order of the squares of the perturbations) from wind to wave; thus his final conclusion is that it could not be exactly steady, although it is steady on an approximation in which squares of perturbations are neglected.

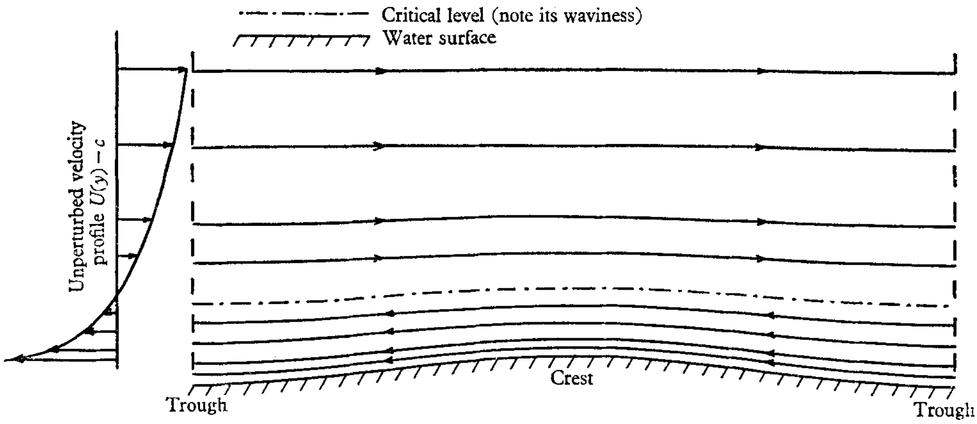

FIGURE 1

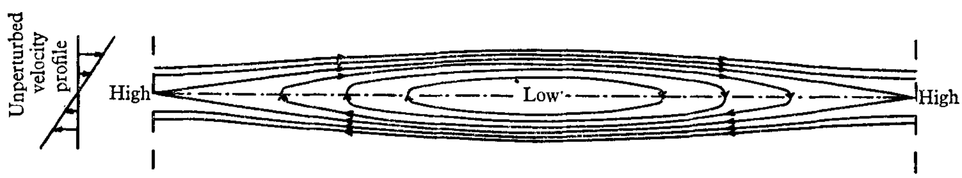

FIGURE 2

FIGURE 1. Airflow relative to a travelling water wave. The figure depicts, in a frame of reference moving to the right with a travelling sinusoidal water wave of velocity c, a sheared airflow, in the direction of the wave, whose unperturbed velocity profile $U(y)-c$ in this frame of reference is as shown, being zero at $y = 0{\cdot}1$ wavelengths. Eight streamlines are shown, representing Miles's first approximation (21), the velocity on each being to a first approximation given by the length of the corresponding arrow on the velocity-profile diagram. The figure exhibits one wavelength only, in the case when the amplitude of the surface elevation is 0·02 wavelengths (that is, 30 % of the maximum possible amplitude). The next approximation to the streamline pattern would be obtained by calculating as in (22) the pressure distribution (low over crests, high over troughs) needed to produce the streamline curvatures of figure 1, and deducing an improved distribution of velocity and streamtube area, by means of the Bernoulli and continuity equations, respectively.

FIGURE 2. Effect of a sinusoidal pressure gradient on a uniformly sheared airflow, in the absence of viscous or turbulent diffusion. Streamlines at equal intervals of the stream function ψ are shown, on the assumption of a velocity gradient at the critical layer as in figure 1, and a pressure distribution there obtained by numerical solution (Miles 1959*a*) of the equations for a logarithmic wind profile linearly perturbed by the surface displacements of figure 1. The application of the pattern of figure 2 to figure 1 might be attempted by bending it so that the dash-dotted critical level takes up the shape which it has in figure 1, but it should be observed that the curvature of the velocity profile assumed in figure 1 would tend to produce compression of the streamlines below the critical level and expansion of those above it, and moreover, that turbulent diffusion would substantially alter the pattern. It would not, however, alter the need for downflow ahead of the crest, and upflow behind it, which results from slow fluid being turned back on approaching the higher-pressure regions over troughs, and must be such that the vortex force, approximately $-\rho U'(y_c)\, v$, balances the sinusoidal pressure gradient. When the unperturbed vorticity decreases with height, this sinsuoidal variation of vertical velocity generates a negative mean vortex force, $-\rho\overline{\omega v}$, capable of extracting air momentum and energy in the correct ratio, c, for transfer to the water wave.

Then Miles's first approximation states that the streamline slope $\theta = v/V(y)$, whose surface value is given to be $\theta_0 \cos(2\pi x/\lambda)$, satisfies everywhere

$$\theta \doteqdot \theta_0 e^{-2\pi y/\lambda} \cos \frac{2\pi x}{\lambda}, \tag{21}$$

in other words, that the streamline pattern is exactly the same (figure 1) as for unsheared flow.

The argument for (21) is in two parts. First, near the water surface, or more specifically for small values of $2\pi y/\lambda$, it gives streamlines which follow closely the shape of the surface streamline, so that the area of a stream-tube remains approximately the same throughout its length, in agreement with the hypothesis of extremely gradual pressure variations in this region which one derives (as in boundary-layer theory) from the argument that pressure variation across the layer must merely be sufficient to balance the centrifugal force associated with the small streamline curvatures. On the other hand, at higher levels, where variations in the unperturbed velocity are relatively small in a distance $\lambda/2\pi$, departures from the streamline-slope variation (21) characteristic of uniform unperturbed velocity should be small.

The former region, where the height is a small fraction of a wavelength, includes the critical level, as discussed elsewhere in this paper. Hence, since the streamlines there follow closely the shape of the surface, and air speed is closely constant along a streamline, it is a good approximation to regard 'critical height' as a fixed height above the *actual* water surface, where air speed is small in a frame of reference moving with the waves. Thus, the critical level is wavy, imitating the undulations of the water surface, as indeed Brooke Benjamin (1959) demonstrated by a co-ordinate transformation, and Miles (1959*a*) attempted to by another, less precisely defined.

Once equation (21) is accepted as a first approximation, the corresponding approximation to the pressure (whose vertical gradient must balance the centrifugal force per unit volume, ρV^2 times the streamline curvature $\partial\theta/\partial x$) can be written down as

$$p = \int_y^\infty \rho V^2(y)\, \theta_0 e^{-2\pi y/\lambda} \left(-\frac{2\pi}{\lambda} \sin \frac{2\pi x}{\lambda}\right) dy. \tag{22}$$

At the critical layer, velocities in the present frame of reference are small, and the pressure and the total pressure (6) are to a close approximation identical. But equation (7) shows that, since the flow is steady when the squares of perturbations are neglected, the gradient of total pressure must be balanced by the vortex force $-\rho\omega v$, so that to this approximation the velocity v at the critical layer is

$$v_0(y_c) \cos \frac{2\pi x}{\lambda} = \frac{\theta_0}{U'(y_c)} \frac{4\pi^2}{\lambda^2} \cos \frac{2\pi x}{\lambda} \int_{y_c}^\infty V^2(y)\, e^{-2\pi y/\lambda}\, dy, \tag{23}$$

in agreement with Miles (1957).

Note that it is the balance between *fluctuating* vortex force and fluctuations of total-pressure gradient, in a frame of reference in which the critical layer is stationary, which fixes the amplitude of upflow fluctuation in the layer; while it is the (second-order) *mean* vortex force produced by these upflow fluctuations (with their capability of transporting vorticity) which causes net transfer of momentum from wind to wave.

As in § 3, it is desirable to study how far the conclusions are valid independently of the supposition that squares of perturbations may be neglected, particularly near the critical layer, which is the only place where that supposition produces singularities. There we have, essentially, a parallel shear flow, including a critical streamline on which the flow velocity is zero, and need to consider the effect upon it of a steady longitudinal sinusoidal pressure variation $p = p_0 \sin(2\pi x/\lambda)$. This problem is easily solved, without any assumption of small perturbations, in the case (treated by many authors from Kelvin onwards) when the vorticity is initially uniform, and figure 2 shows the well-known streamlines, calculated from the stream function

$$\psi = \tfrac{1}{2}U'(y_c)\,(y-y_c)^2 + \frac{p_0 \sin(2\pi x/\lambda)\cosh(2\pi y/\lambda)}{\rho U'(y_c)}; \tag{24}$$

oscillations of vertical velocity are produced at $y = y_c$ exactly equal to those calculated by balancing vortex force and pressure gradient.

The flow field when the undisturbed vorticity is not uniform, but rather decreases upwards, is probably not qualitatively different from that of figure 2. The pressure variation must similarly reduce slow-moving fluid (just above, or below, the critical height) to rest and cause it to move vertically before turning back on its tracks.† Admittedly, in the total absence of diffusion, viscous or turbulent, the closed streamlines shown in figure 2 would fail (as noted in § 3) to produce any mean vortex force $\rho\overline{\omega v}$; however, any kind of diffusion makes it possible for fluid before entering on its region of maximum vertical movement in either direction to have acquired vorticity characteristic of its surroundings, and then $\rho\overline{v\omega}$ is positive and close to that calculated in § 3.

On the other hand, the logarithmic infinity in horizontal velocity, indicated by the small-perturbation theory, is not supported by more exact considerations, nor is the associated term proportional to $1/(y-y_c)$ in the disturbance vorticity. The true change in vorticity remains finite, and bounded by the product of $-U''(y_c)$ with the greatest vertical displacement of fluid particles, which in figure 2 is approximately $4(p_0/\rho)^{\frac{1}{2}}/U'(y_c)$.‡ Thus it remains small, and only its

† The 'sheltering' theory of Jeffreys (1925), which supposed that wind can undergo boundary-layer separation in the lee of wave crests, was largely demolished by Ursell (1956), who observed that momentum transfer in airflow over even solid wavy surfaces is experimentally an order of magnitude less than Jeffreys requires; while, for waves on water, propagated in the direction of the wind, the air near the surface moves forward more slowly than do the wave crests, so that separation on the windward side might, rather, be expected! The true situation, as figure 2 indicates, is dominated not by conditions at the water surface but at the critical layer, and also is far more symmetrical, since air moving forward into the lee of a crest does, at that height, suffer reversed flow, but so does air falling back to the windward of a crest; while the surface pressure defect ahead of crests is reinforced by a pressure excess behind them. It is however, possible, that for wave amplitudes nearer the maximum than that of figure 1 the region of reversed flow widens to include the water surface itself, which would increase the resemblance to conventional flow separation.

‡ Incidentally, half this quantity agrees well with the 'critical-layer thickness' δ calculated by the rough argument leading to (18).

proportionality to the square root of the disturbance causes it to be infinite on a purely linear theory. Indeed, there are conditions of more extensive diffusion, discussed in § 3, under which it is smaller still; these would modify substantially the 'cat's eye' pattern of figure 2, while retaining, however, the vertical movements required to balance the sinusoidal pressure gradient with a sinusoidal vortex force.

It is natural to ask, finally, how the disturbances at the critical layer, which lead to local loss of momentum and energy, contrive to transfer these to the water wave. The answer is somewhat as follows.

Undulations like those of figure 1 produce maximum streamline displacements, and hence also maximum vorticity perturbations, over crests and troughs. Such vorticity perturbations evidently induce a velocity field whose horizontal component has maxima over crests and troughs and whose vertical component has maxima over nodes. Such an induced velocity field cannot alter the property of the streamline pattern, that it has maximum displacement over crests and troughs; nor will this conclusion be altered after indefinite repetition of the argument; moreover, in such a pattern the pressure must be in perfect antiphase with the surface elevation, and no energy can be transferred to the waves.

However, conditions at the critical layer form an exception to what has just been described; there, as we have seen, the vertical velocities, with maxima over nodes, generate local concentrations of excess vorticity with maxima over nodes. These can only induce a velocity field whose horizontal component has maxima over nodes, and whose vertical component has maxima over crests and troughs; this produces a redistribution of all the rest of the vorticity; this induces another velocity field in the same phase, which produces a further redistribution, and so on; the details are complex, but the essential point is that at each stage of *this* process the horizontal components of velocity, and hence also the pressures, have maxima over nodes. In particular, at the surface, a component of pressure lagging 90° behind the surface elevation results in this way from the vorticity displacements at the critical layer; its magnitude, however, is found far more easily from the arguments about momentum loss given in § 3.

These explanations of how the long-range action of vorticity concentrated at the critical layer, out of phase with the main vorticity distribution, generates at the surface momentum transfer to the wave, do not, of course, deny that across every level, between the critical layer and the surface, transport of momentum must occur; in other words, that $\rho\overline{uv}$, the Reynolds stress formed from the sinusoidally varying velocities u and v (as opposed to any turbulent components superimposed on them) must be negative. Indeed, this Reynolds stress is the integral from y to ∞ of the mean vortex force $-\rho\overline{\omega v}$ and must, in so far as that force shows a delta-function behaviour at the critical level, be zero above and constant below that level. Lin (1955) and others have drawn attention to this behaviour of $\rho\overline{uv}$ in connexion with the theory of hydrodynamic stability, but have not used its derivative, the mean vortex force, which seems to help in the understanding of at least the present problem.

5. Conclusion

Miles's mechanism of water-wave amplification by wind, which is quantitatively supported, as noted in § 1, by the observations on directional spectra by Longuet-Higgins *et al.* (1962), can be described concisely as follows.

Sinusoidal travelling waves on water perturb air flowing over them (in their direction of propagation) by undulations, in parallel with those in the water surface. These produce an air pressure distribution, which at any level is greatest over troughs and least over crests. Just above the critical level, the air is moving a little faster than the wave velocity, but, as it creeps forward from over a crest to over a trough, it is turned back by the higher pressure, moves down to below the critical layer and returns towards the crest. Quantitatively, the downflow ahead of the crest must be such as to provide a vortex force $-\rho\omega v$ balancing the pressure gradient; upflow is similarly produced behind the crest.

If, now, in the undisturbed airflow, the vorticity decreases with height, as is characteristic of turbulent flow, then the upflow behind the crest carries with it vorticity in excess of the unperturbed value characteristic of its new surroundings, while the downflow ahead of the crest carries a defect of vorticity. The mean vortex force $-\rho\overline{\omega v}$ is accordingly negative, and reduces the mean momentum at the critical layer, where energy and momentum are lost in the correct ratio, c, for transfer to the water-wave motion. The rate of this energy transfer varies as the square of the wave amplitude, and hence as the wave energy itself, which therefore grows exponentially.

The mechanism is effective only for small y_c/λ. This is because the undulations of the airflow decay with height like $\exp(-2\pi y/\lambda)$; hence, the pressure gradients at the critical layer, and the upflow velocities they produce, are roughly proportional to $\exp(-2\pi y_c/\lambda)$; accordingly, the momentum transfer, which depends on the square of the upflow velocities, is roughly proportional to $\exp(-4\pi y_c/\lambda)$, as noted in § 1. We have seen that for such small y_c/λ the 'critical level', where air velocity equals wave velocity, is more closely at a fixed height y_c above the wavy water surface, than above the mean water surface, and must be thought of, therefore, as imitating the undulations of the surface.

When the airflow makes a non-zero angle θ to the direction of wave propagation, the same arguments apply provided that the unperturbed velocity is replaced, throughout, by its component in the direction of wave propagation. The critical layer is thus defined so that $U(y_c)\cos\theta = c$; this ensures that fluid at that level moves so that it can keep up with a given wave crest or trough, as in the above discussion.

We may note, finally, that experiments are conceivable in which only waves in the direction of the wind are permitted to be present; in these, the loss of momentum only in a rather narrow band of heights might be expected to lead to a peculiarly shaped velocity profile. However, in the case of a natural wind, momentum is lost at a great variety of heights, varying with the angle θ between the wind and the direction of propagation of the parts of the water-wave spectrum to which momentum at the height in question is transferred. In this case, therefore, no special peculiarities of velocity profile are to be expected.

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